Team Solutions Packet

1. Find the integer n such that
$$n + \lfloor \sqrt{n} \rfloor + \left\lfloor \sqrt{\sqrt{n}} \right\rfloor = 2017.$$
Here, as usual, $\lfloor \cdot \rfloor$ denotes the floor function.

   Proposed by Patrick Lin

   Solution. Noting that $44 < \sqrt{2017} < 45$ and $6 < \sqrt{\sqrt{2017}} < 7$ gives us a lower bound on n of $2017-45-7 = 1965$. Now remark that since $\lfloor \sqrt{1965} \rfloor = \lfloor \sqrt{1968} \rfloor = 44$, $n = 1967$ is achievable but $n = 1968$ is not, for an answer of $\boxed{1967}$.

2. Suppose x, y, and z are nonzero complex numbers such that $(x+y+z)(x^2+y^2+z^2) = x^3+y^3+z^3$. Compute
$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$

   Proposed by David Altizio

   Solution. Let $S_1 = x+y+z$, $S_2 = xy+yz+zx$, and $S_3 = xyz$. Note that $x^2+y^2+z^2 = S_1^2 - 2S_2$ and
$$x^3+y^3+z^3 = (x+y+z)(x^2+y^2+z^2-xy-yz-zx) + 3xyz = S_1^3 - 3S_1S_2 + 3S_3.$$
Thus the condition becomes
$$S_1(S_1^2 - 2S_2) = S_1^3 - 3S_1S_2 + 3S_3 \qquad \Longrightarrow \qquad S_1S_2 = 3S_3.$$
Hence $(x+y+z)(xy+yz+zx) = 3xyz$, and so dividing both sides by xyz gives the desired answer of $\boxed{3}$.

3. Suppose Pat and Rick are playing a game in which they take turns writing numbers from $\{1, 2, \ldots, 97\}$ on a blackboard. In each round, Pat writes a number, then Rick writes a number; Rick wins if the sum of all the numbers written on the blackboard after n rounds is divisible by 100. Find the minimum positive value of n for which Rick has a winning strategy.

   Proposed by Andrew Kwon

   Solution. Suppose that Rick has a winning strategy for some value of n and let S_i denote the sum of the numbers on the board modulo 100 after round i. At the end of the n^{th} round, the sum of all the numbers written on the board must be $0 \pmod{100}$. Now, consider the state of the game at the end of the $(n-1)^{\text{th}}$ round. We claim that $S_{n?1} \equiv 2 \pmod{100}$ in order for Rick to be guaranteed to achieve $S_n?0 \pmod{100}$. If $S_{n?1} \equiv 2 \pmod{100}$, then no matter what value k Pat chooses, Rick can write $98?k$ and win the game. On the other hand, if $S_{n?1} \equiv \ell \pmod{100}$ where $\ell \neq 2$ then Pat can choose
$$k = \begin{cases} 1 & \text{if } \ell = 0, 1 \\ 100 - \ell & \text{otherwise,} \end{cases}$$
and in these cases Rick cannot guarantee a win. Proceeding inductively, we find that $S_{n-i} \equiv 2i \pmod{100}$ is necessary in order for Rick to win. Evidently $S_0 \equiv 0 \pmod{100}$, and so $S_0 \equiv 2n \equiv 0 \pmod{100}$. The minimal n satisfying this is $n = \boxed{50}$.

4. Say a positive integer $n > 1$ is *twinning* if $p - 2 \mid n$ for every prime $p \mid n$. Find the number of twinning integers less than 250.

   Proposed by Andrew Kwon

   Solution. We case on the largest prime factor q of n, noting that n cannot be divisible by 2. We must have $q < 17$, as $17 \cdot 15 = 255$, while $q = 13$ is also impossible because this implies that n is divisible by $13 \cdot 11 \cdot 9$.

- $q = 11$: We have $11 \cdot 9 \mid n$, while no other prime factors are possible, so $n = 99$ is one possibility in this case.
- $q = 7$: Again, $7 \cdot 5 \cdot 3 \mid n$, and again no other prime factors are possible, so $n = 105$ is one possibility in this case.
- $q = 5$: We must have $5 \cdot 3 \mid n$, while n can only contain factors of 5 and 3. We find $5 \cdot 3, 5^2 \cdot 3, 5 \cdot 3^2, 5^2 \cdot 3^2, 5 \cdot 3^3$ are all twinning, so there are 5 possibilities in this case.
- $q = 3$: The powers of 3, namely $3, 9, 27, 81, 243$ are all twinning.

We conclude there are $\boxed{12}$ twinning numbers less than 250.

5. We have four registers, R_1, R_2, R_3, R_4, such that R_i initially contains the number i for $1 \le i \le 4$. We are allowed two operations:

    - Simultaneously swap the contents of R_1 and R_3 as well as R_2 and R_4.
    - Simultaneously transfer the contents of R_2 to R_3, the contents of R_3 to R_4, and the contents of R_4 to R_2. (For example if we do this once then $(R_1, R_2, R_3, R_4) = (1, 4, 2, 3)$.)

    Using these two operations as many times as desired and in whatever order, what is the total number of possible outcomes?

    Proposed by Cody Johnson

    Solution. We're looking for the number of distinct permutations generated by the two permutations $\pi_1 := (1 \mapsto 3, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 2)$ and $\pi_2 := (1 \mapsto 1, 2 \mapsto 4, 3 \mapsto 2, 4 \mapsto 3)$. Note that each of these two permutations are even, so we can only generate even permutations from them. Therefore, we can generate at most $4!/2 = 12$ permutations. Furthermore, we can generate the following 12 permutations, so the answer is $\boxed{12}$:

    $$\begin{array}{cccc} (1,2,3,4) & (1,4,2,3) & (1,3,4,2) & (4,2,1,3) \\ (4,3,2,1) & (4,1,3,2) & (3,2,4,1) & (3,1,2,4) \\ (3,4,1,2) & (2,4,3,1) & (2,1,4,3) & (2,3,1,4) \end{array}$$

6. George is taking a ten-question true-false exam, where the answer key has been selected uniformly at random; however, he doesn't know any of the answers! Luckily, a friend has helpfully hinted that no two consecutive questions have true as the correct answer. If George takes the exam and maximizes the expected number of questions he gets correct, how many of his answers are expected to be right?

    Proposed by Patrick Lin

    Solution. We claim George's strategy is to answer false on every question; by linearity of expectation it suffices to show that the answer to each question will be false with probability greater than $1/2$. Let D be any assignment of answers satisfying the problem conditions, i.e. no two consecutive answers are T (here, T is true and F is false), and consider the k-th answer, for some k. If it is T, then we can always change it to F and still have a valid assignment; if it is F, however, there exist assignments where we cannot change it to T, which is when an adjacent answer is already T. Hence there are more assignments where the k-th answer is F than T, and so the probability that some answer is F is greater than $1/2$.

    Now it remains to determine the expected number of F's that appear in a randomly chosen assignment of answers with no consecutive T's. Note that if an assignment begins with a T the next answer must be an F and otherwise if it begins with F the next answer is unrestricted, and so by straightforward recursion there are F_{n+2} possible assignments to n questions, where F is the Fibonacci sequence. Let t_n be the total number of T's over all assignments to n questions, so $t_0 = 0$ and $t_1 = 1$. By a similar argument, we have the recursion

    $$t_n = t_{n-1} + (t_{n-2} + F_n),$$

    since there are t_{n-1} total T's that appear among all assignments to n questions that begin with F, and $t_{n-2} + F_n$ total that appear among those that begin with a T, as there are t_{n-2} total T's in the $n - 2$

questions at the end and F_n at the front, one per assignment. This yields $t_{10} = 420$ and $F_{12} = 144$, so the expected number of T's is $\frac{420}{144} = \frac{35}{12}$.

The answer is hence $10 - \frac{35}{12} = \boxed{\frac{85}{12}}$.

7. Define $\{p_n\}_{n=0}^{\infty} \subset \mathbb{N}$ and $\{q_n\}_{n=0}^{\infty} \subset \mathbb{N}$ to be sequences of natural numbers as follows:

   - $p_0 = q_0 = 1$;
   - For all $n \in \mathbb{N}$, q_n is the smallest natural number such that there exists a natural number p_n with $\gcd(p_n, q_n) = 1$ satisfying
   $$\frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} < \sqrt{2}.$$

   Find q_3.

   Proposed by David Altizio

   Solution. Shift the sequence down by 1, so that $p_0 = 0$ and the upper bound in question is $\sqrt{2} - 1$; this makes the arithmetic a little bit easier. It is not hard to see that $p_1/q_1 = 1/3$ and $p_2/q_2 = 2/5$ are the first two terms of this sequence; the difficult part lies in extending this further.

   Write $p_3 = (2q_3 + r_3)/5$, where $0 < q_3 < 5$. The condition that $p_3/q_3 < \sqrt{2} - 1$ is equivalent to
   $$\left(\frac{p_3 + q_3}{q_3}\right)^2 < 2 \quad \Longrightarrow \quad q_3^2 > p_3^2 + 2p_3q_3.$$

   Making the substitution yields
   $$\begin{aligned} q_3^2 &> \left(\frac{2q_3 + r_3}{5}\right)^2 + 2\left(\frac{2q_3 + r_3}{5}\right) q_3 \\ &= \frac{24}{25}q_3^2 + \frac{14}{25}q_3r_3 + \frac{1}{25}r_3^2 \\ \Longrightarrow \quad q_3^2 &> 14q_3r_3 + r_3^2. \end{aligned}$$

   Note that we necessarily need $q_3 > 14$, since otherwise the RHS will be strictly bigger. In addition, in order to minimize q_3, we need $r_3 = 1$. The smallest integer such that this is the case is $q_3 = 17$. Indeed, we find that this works with $p_3 = 7$, so our answer is $\boxed{17}$.

8. Alice and Bob have a fair coin with sides labeled C and M, and they flip the coin repeatedly while recording the outcomes; for example, if they flip two C's then an M, they have CCM recorded. They play the following game: Alice chooses a four-character string $\mathcal{A}$, then Bob chooses two distinct three-character strings $\mathcal{B}_1$ and $\mathcal{B}_2$ such that neither is a substring of $\mathcal{A}$. Bob wins if $\mathcal{A}$ shows up in the running record before either $\mathcal{B}_1$ or $\mathcal{B}_2$ do, and otherwise Alice wins. Given that Alice chooses $\mathcal{A} = CMMC$ and Bob plays optimally, compute the probability that Bob wins.

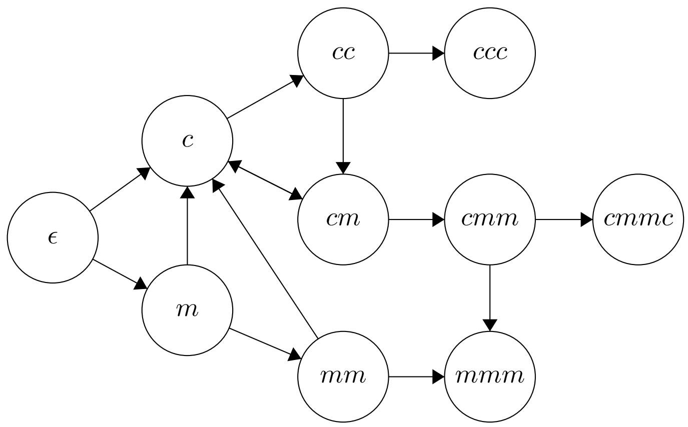
   Proposed by Patrick Lin

   Solution. (Sketch) Bob chooses $B_1, B_2 = CCC, MMM$, which gives him a winning probability of $\boxed{\frac{21}{80}}$.

   Solution. First we give an intuitive explanation that $\mathcal{B}_1, \mathcal{B}_2 = CCC, MMM$. Observe that an ideal pair $(\mathcal{B}_1, \mathcal{B}_2)$ satisfies the qualities that

   - almost satisfying one string before "falling off" leads to a suffix that is still fairly far from completing the other string,
   - "falling off" from completing $\mathcal{A} = CMMC$ will, in most cases, lead to a suffix that is also fairly far from $\mathcal{B}_1$ or $\mathcal{B}_2$.

Considering these two criteria makes CCC and MMM a clear guess, especially since it is difficult to "confuse" one string for another as the coins are flipped (as an aside, the second-best pair is to choose CCC and MCC). Once we have $\mathcal{B}_1$ and $\mathcal{B}_2$, we can construct a Markov chain as shown below, where each arrow is taken with probability $\frac{1}{2}$.



Identifying C as the probability that Bob wins given that we are in state C and that $CCC = MMM = 0$ and $CMMC = 1$, we have the system of equations

$$CMM = \frac{1}{2} \tag{1}$$

$$CM = \frac{1}{2}CMM + \frac{1}{2}C \tag{2}$$

$$MM = \frac{1}{2}C \tag{3}$$

$$CC = \frac{1}{2}CM \tag{4}$$

$$M = \frac{1}{2}C + \frac{1}{2}MM \tag{5}$$

$$C = \frac{1}{2}CC + \frac{1}{2}CM. \tag{6}$$

Substituting these equations from the top down yields $C = \frac{3}{10}$ and $M = \frac{3}{4}C = \frac{9}{40}$, and so the chance that Bob wins is

$$\frac{1}{2}C + \frac{1}{2}M = \boxed{\frac{21}{80}}.$$

9. Circles ω_1 and ω_2 are externally tangent to each other. Circle Ω is placed such that ω_1 is internally tangent to Ω at X while ω_2 is internally tangent to Ω at Y. Line ℓ is tangent to ω_1 at P and ω_2 at Q and furthermore intersects Ω at points A and B with $AP < AQ$. Suppose that $AP = 2$, $PQ = 4$, and $QB = 3$. Compute the length of line segment $\overline{XY}$.

Proposed by David Altizio

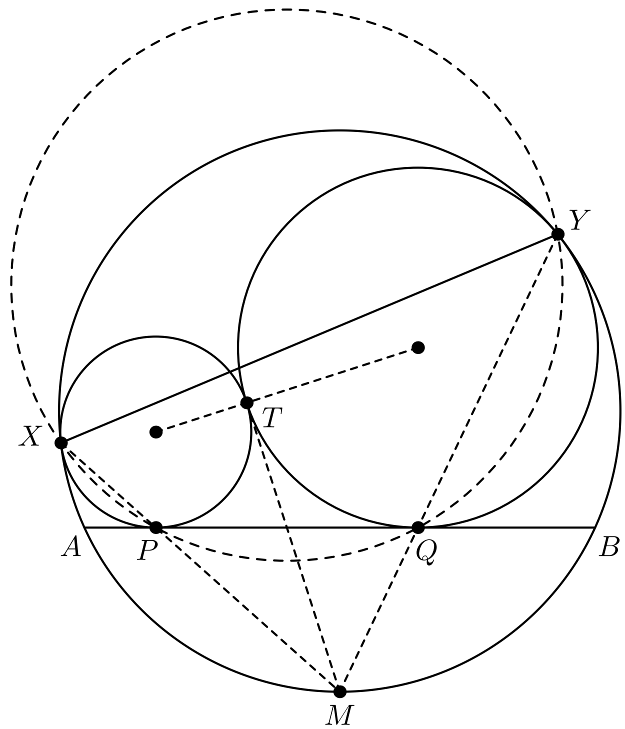
Solution. First, recall by homothety that $M = XP \cap YQ$ is the midpoint of minor arc $\widehat{AB}$. This means that

$$\angle XPA = \frac{\widehat{AX} + \widehat{MB}}{2} = \frac{\widehat{AX} + \widehat{MA}}{2} = \angle XYQ,$$

whence $XPQY$ is a cyclic quadrilateral. Now M is the radical center of ω_1, ω_2, and $\odot(XPQY)$, so in particular if $T = \omega_1 \cap \omega_2$, then MT is the common internal tangent of the two circles.

Set $r = MA = MB$, and define $D = \overline{MT} \cap \overline{AB}$. Note that a bit of angle chasing yields $\triangle MAP \sim \triangle MXA$, so

$$MA^2 = MP \cdot MX = MT^2 \quad \Longrightarrow \quad MT = r.$$



Thus $DP = DQ = DT = 2$ and $MD = r - 2$. Now Stewart's Theorem on $\triangle MAB$ with cevian $\overline{MD}$ gives us the value of r directly, by

$$4 \cdot 5 \cdot 9 + 9(r-2)^2 = 4r^2 + 5r^2 \implies r = 6.$$

Finally, if we let $x = MP$ and $y = MQ$ now, we have $PX = \frac{AP \cdot PB}{PM} = \frac{14}{x}$, and similarly $QY = \frac{AQ \cdot QB}{MQ} = \frac{18}{y}$. Thus

$$x\left(x + \frac{14}{x}\right) = y\left(y + \frac{18}{y}\right) = r^2 = 36,$$

so $x = \sqrt{22}$ and $y = \sqrt{18}$ and $XY = \frac{4}{x}MY = \boxed{\frac{24}{\sqrt{11}}}$.

Remark. An alternate way to prove the geometric results listed above is through inversion. As before, let $r = MA = MB$. The inversion at M with radius r swaps line AB and Ω, hence fixing ω_1 and ω_2 and thus swapping $X \mapsto P$ and $Y \mapsto Q$. This means that MXP, MYQ collinear with $r^2 = MP \cdot MX = MQ \cdot MY$, and so $XPQY$ is a cyclic quadrilateral. In particular M is the radical center of ω_1, ω_2, and $\odot(XPQY)$, so if $T = \omega_1 \cap \omega_2$, then $MT = r$.

10. The polynomial $P(x) = x^3 - 6x - 2$ has three real roots, α, β, and γ. Depending on the assignment of the roots, there exist two different quadratics Q such that the graph of $y = Q(x)$ pass through the points (α, β), (β, γ), and (γ, α). What is the larger of the two values of $Q(1)$?

Proposed by David Altizio

Solution. Let $Q(x) = ax^2 + bx + c$ for some real numbers a, b, and c, so that

$$\begin{cases} a\alpha^2 + b\alpha + c &= \beta, \\ a\beta^2 + b\beta + c &= \gamma, \\ a\gamma^2 + b\gamma + c &= \alpha. \end{cases}$$

The main idea is to manipulate the equations in such a way that various instances of Vieta's Formulas can be used to give a system of equations.

As a preliminary, compute $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\gamma\alpha=-6$, $\alpha\beta\gamma=2$, $\alpha^2+\beta^2+\gamma^2=12$, and $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma=6$. Note that adding the three equations together yields

$$a(\alpha^2+\beta^2+\gamma^2)+b(\alpha+\beta+\gamma)+3c=\alpha+\beta+\gamma \qquad \Longrightarrow \qquad c=-4a.$$

Multiplying the first equation by α, the second equation by β, and the third equation by γ and adding yields

$$a(\alpha^3+\beta^3+\gamma^3)+b(\alpha^2+\beta^2+\gamma^2)+c(\alpha+\beta+\gamma)=\alpha\beta+\beta\gamma+\gamma\alpha \qquad \Longrightarrow \qquad a+2b=-1.$$

Getting the third equation is a bit tougher. Note that multiplying the first equation by β, the second equation by γ, and the third by α and adding yields

$$a(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha)+b(\alpha\beta+\beta\gamma+\gamma\alpha)+c(\beta+\gamma+\alpha)=\beta^2+\gamma^2+\alpha^2 \qquad \Longrightarrow \qquad a(\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha)-6b=12.$$

Similarly, multiplying the first equation by $\beta\gamma$, the second by $\gamma\alpha$, and the third by $\alpha\beta$ and adding gives

$$a(\alpha^2\beta\gamma+\alpha\beta^2\gamma+\alpha\beta\gamma^2)+3b\alpha\beta\gamma+c(\beta\gamma+\gamma\alpha+\alpha\beta)=\beta^2\gamma+\gamma^2\alpha+\alpha^2\beta \qquad \Longrightarrow \qquad 6b-6c=\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha.$$

Hence substitution yields

$$a(6b-6c)-6b=12,$$

or $a(b-c)-b=2$.

As a result, the manipulations above lead to the system of equations

$$\begin{cases} c &= -4a, \\ a+2b &= -1, \\ a(b-c)-b &= 2. \end{cases}$$

Multiplying the third equation by 2 and substituting for a from the first and second equations yields

$$2a(b-c)-2b=a(-1-a+8a)-(-1-a)=7a^2+1=4,$$

so $a=\pm\sqrt{\frac{3}{7}}=\pm\frac{\sqrt{21}}{7}$. Finally, remark that

$$2(a+b+c)=a+(a+2b)+2c=a-1-8a=-7a-1=\mp\sqrt{21}-1,$$

so the requested answer is $\boxed{\frac{\sqrt{21}-1}{2}}$.