CMIMC 2016

Number Theory Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. All answers are integers.
7. If you believe that the test contains an error, submit your protest in writing to Porter 100.

Number Theory

1. David, when submitting a problem for CMIMC, wrote his answer as $100\frac{x}{y}$, where x and y are two positive integers with $x < y$. Andrew interpreted the expression as a product of two rational numbers, while Patrick interpreted the answer as a mixed fraction. In this case, Patrick's number was exactly double Andrew's! What is the smallest possible value of $x + y$?

2. Let a_1, a_2, ... be an infinite sequence of integers such that k divides $\gcd(a_{k-1}, a_k)$ for all $k \geq 2$. Compute the smallest possible value of $a_1 + a_2 + \cdots + a_{10}$.

3. How many pairs of integers (a, b) are there such that $0 \leq a < b \leq 100$ and such that $\frac{2^b - 2^a}{2016}$ is an integer?

4. For some positive integer n, consider the usual prime factorization

$$n = \prod_{i=1}^{k} p_i^{e_i} = p_1^{e_1} p_2^{e_2} \dots p_k^{e_k},$$

where k is the number of primes factors of n and p_i are the prime factors of n. Define $Q(n), R(n)$ by

$$Q(n) = \prod_{i=1}^{k} p_i^{p_i} \text{ and } R(n) = \prod_{i=1}^{k} e_i^{e_i}.$$

For how many $1 \leq n \leq 70$ does $R(n)$ divide $Q(n)$?

5. Determine the sum of the positive integers n such that there exist primes p, q, r satisfying $p^n + q^2 = r^2$.

6. Define a *tasty residue* of n to be an integer $1 < a < n$ such that there exists an integer $m > 1$ satisfying

$$a^m \equiv a \pmod{n}.$$

Find the number of tasty residues of 2016.

7. Determine the smallest positive prime p which satisfies the congruence

$$p + p^{-1} \equiv 25 \pmod{143}.$$

Here, p^{-1} as usual denotes multiplicative inverse.

8. Given that

$$\sum_{x=1}^{70} \sum_{y=1}^{70} \frac{x^y}{y} = \frac{m}{67!}$$

for some positive integer m, find $m \pmod{71}$.

9. Compute the number of positive integers $n \leq 50$ such that there exist distinct positive integers a, b satisfying

$$\frac{a}{b} + \frac{b}{a} = n\left(\frac{1}{a} + \frac{1}{b}\right).$$

10. Let $f : \mathbb{N} \mapsto \mathbb{R}$ be the function

$$f(n) = \sum_{k=1}^{\infty} \frac{1}{\operatorname{lcm}(k, n)^2}.$$

It is well-known that $f(1) = \frac{\pi^2}{6}$. What is the smallest positive integer m such that $m \cdot f(10)$ is the square of a rational multiple of π?