Geometry Solutions Packet

1. Let ABC be a triangle with $\angle BAC = 117^\circ$. The angle bisector of $\angle ABC$ intersects side AC at D. Suppose $\triangle ABD \sim \triangle ACB$. Compute the measure of $\angle ABC$, in degrees.

   Proposed by David Altizio

   Solution. Note that $\angle ABD = \angle ACB$ by this similarity, so $\angle ABC = 2\angle ACB$. Letting the measure of $\angle ACB$ in degrees be θ, we have

   $$\theta + 2\theta + 117 = 180 \qquad \Longrightarrow \qquad \theta = 21$$

   and so $\angle ABC = \boxed{42^\circ}$.

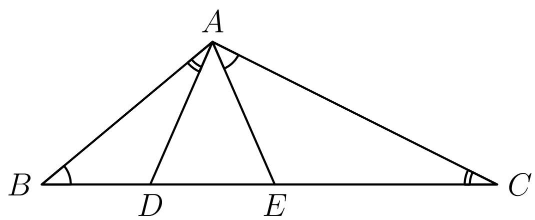
2. Triangle ABC has an obtuse angle at $\angle A$. Points D and E are placed on $\overline{BC}$ in the order B, D, E, C such that $\angle BAD = \angle BCA$ and $\angle CAE = \angle CBA$. If $AB = 10$, $AC = 11$, and $DE = 4$, compute BC.

   Proposed by David Altizio

   Solution. For simplicity let $BC = a$, $CA = b$, and $AB = c$. Note that $\triangle ABD \sim \triangle CBA$, so $BD = \frac{c^2}{a}$. Similarly, $CE = \frac{b^2}{a}$, so

   $$DE = a - \frac{c^2}{a} - \frac{b^2}{a} = a - \frac{b^2 + c^2}{a} = a - \frac{221}{a} = 4.$$

   Solving this quadratic yields $a = \boxed{17}$.

   Remark: The obtuse condition is necessary in order for the points B, D, E, and C to actually be in that order; this is because $\angle BAD + \angle CAE < 90^\circ < \angle BAC$. Indeed, a triangle with side lengths 10, 11, and 17 has an obtuse angle with degree measure $\approx 108^\circ$.

3. In acute triangle ABC, points D and E are the feet of the angle bisector and altitude from A respectively. Suppose that $AC - AB = 36$ and $DC - DB = 24$. Compute $EC - EB$.

   Proposed by David Altizio

   Solution. Let $AC = x$ and $BC = y$. Note that by Angle Bisector Theorem,

   $$\frac{DC}{AC} = \frac{DB}{AB} = \frac{DC - DB}{AC - AB} = \frac{2}{3}.$$

   Thus $DC = \frac{2}{3}x$ and $DB = \frac{2}{3}y$. Now note that by Pythagorean Theorem, $EC^2 - EB^2 = AC^2 - AB^2$. This means that

   $$(EC - EB)(EC + EB) = (AC - AB)(AC + AB) \qquad \Longrightarrow \qquad (EC - EB) \cdot \frac{2}{3}(x + y) = 36(x + y).$$

   Simplification yields $EC - EB = 36 \cdot \frac{3}{2} = \boxed{54}$.

   Remark: This generalizes to the intersting identity $(DC - DB)(EC - EB) = (AC - AB)^2$.

4. Let $\mathcal{S}$ be the sphere with center $(0,0,1)$ and radius 1 in $\mathbb{R}^3$. A plane $\mathcal{P}$ is tangent to $\mathcal{S}$ at the point (x_0, y_0, z_0), where x_0, y_0, and z_0 are all positive. Suppose the intersection of plane $\mathcal{P}$ with the xy-plane is the line ℓ with equation $2x + y = 10$ in xy-space. What is z_0?

   Proposed by David Altizio

   Solution. Let O be the origin, C the center of $\mathcal{S}$, and T the point of tangency of $\mathcal{S}$ with $\mathcal{P}$. Denote by P the projection of O onto ℓ, and consider the cross-section of this figure passing through P perpendicular to ℓ. Then $\mathcal{S}$ becomes a circle ω with radius 1, and OP is tangent to ω. It is intuitively clear that PT is the other tangent to ω in this cross section; we continue with the computation and then prove this fact afterwards.

   Note that the line ℓ cuts a right triangle with side lengths 5 and 10 in the xy-plane. Thus, the length of the altitude from O to ℓ is $\frac{5\cdot 10}{\sqrt{5^2+10^2}} = 2\sqrt{5}$, i.e. $OP = 2\sqrt{5}$. Thus Pythagorean Theorem gives $CP = \sqrt{1^2 + (2\sqrt{5})^2} = \sqrt{21}$. Now let $\angle OPC = \theta$. Compute

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{1}{\sqrt{21}}\right)\left(\frac{2\sqrt{5}}{\sqrt{21}}\right) = \frac{4\sqrt{5}}{21}.$$

   Thus

$$\sin 2\theta = \frac{z_0}{PT} \quad\Longrightarrow\quad z_0 = PT\sin 2\theta = 2\sqrt{5}\cdot\frac{4\sqrt{5}}{21} = \boxed{\frac{40}{21}}.$$

   It remains to prove the assertion at the end of the first paragraph. To do this, we use the formal definition of a plane. Recall that for any point A and vector $\vec{n}$, the set of all points B such that $\overrightarrow{AB}$ is perpendicular to $\vec{n}$ forms a plane. Thus any plane can be specified by a point in said plane a vector normal to the plane. (Of course, this normal vector is not unique!)

   With this, we can formally prove the above statement. Let $\mathcal{Q}$ denote the plane which forms the cross-section defined above; it suffices to show that T lies in $\mathcal{Q}$. Note that since $\mathcal{P}$ is tangent to $\mathcal{S}$ at T, we know that $\overrightarrow{TC}$ is normal to $\mathcal{P}$. Since $\ell \in \mathcal{P}$, we deduce that $\ell \perp \overrightarrow{TC}$. But remark that $\overrightarrow{OC}$ is normal to the xy-plane, which ℓ lies in, so $\overrightarrow{OC} \perp \ell$. Combining this with the fact that $OP \perp \ell$ by the definition of projection gives that ℓ is normal to the entire plane $\mathcal{Q}$. Thus $T \in \mathcal{Q}$ as desired.

5. Two circles ω_1 and ω_2 are said to be *orthogonal* if they intersect each other at right angles. In other words, for any point P lying on both ω_1 and ω_2, if ℓ_1 is the line tangent to ω_1 at P and ℓ_2 is the line tangent to ω_2 at P, then $\ell_1 \perp \ell_2$. (Two circles which do not intersect are not orthogonal.)

   Let $\triangle ABC$ be a triangle with area 20. Orthogonal circles ω_B and ω_C are drawn with ω_B centered at B and ω_C centered at C. Points T_B and T_C are placed on ω_B and ω_C respectively such that AT_B is tangent to ω_B and AT_C is tangent to ω_C. If $AT_B = 7$ and $AT_C = 11$, what is $\tan\angle BAC$?

   Proposed by David Altizio

   Solution. We first proceed with a lemma.

   LEMMA: If ω_1 and ω_2 are orthogonal circles with radii r_1 and r_2 respectively, and d is the distance between the centers of these two circles, then

$$r_1^2 + r_2^2 = d^2.$$

   Proof. Let P be a point of intersection of ω_1 and ω_2, and let O_1 and O_2 denote the centers of ω_1 and ω_2 respectively. Note that by the definition of tangency, PO_1 is perpendicular to the line tangent to ω_1 at P. But recall that by the definition of orthogonal circles, the tangents to ω_1 and ω_2 passing through P are perpendicular. Hence $PO_1 \perp PO_2$, and the desired follows from Pythagorean Theorem. ■

   Let r_B and r_C denote the radii of ω_B and ω_C respectively. Note that by Pythagorean Theorem,

$$AT_B^2 = AB^2 - r_B^2 \qquad \text{and} \qquad AT_C^2 = AC^2 - r_C^2.$$

Adding these together yields

$$AT_B^2 + AT_C^2 = AB^2 + AC^2 - (r_B^2 + r_C^2)$$
$$= AB^2 + AC^2 - BC^2 = 2(AB)(AC)\cos\angle BAC,$$

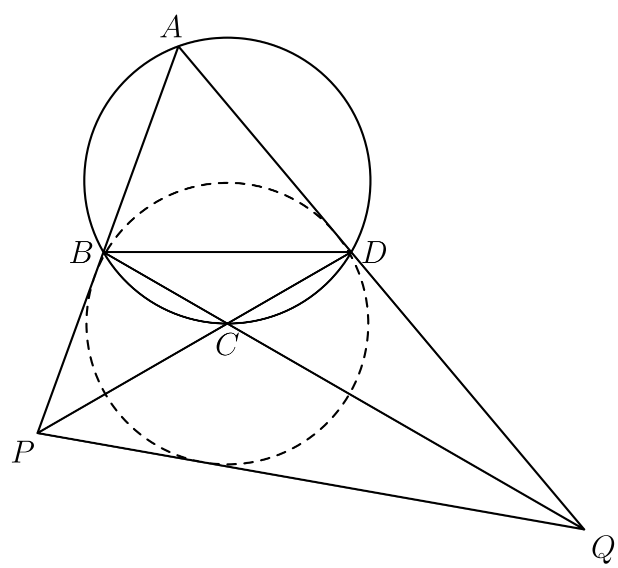
where the last step follows from Law of Cosines. Combined with $\frac{1}{2}(AB)(AC)\sin\angle BAC = [ABC]$, it follows that

$$\tan\angle BAC = \frac{\sin\angle BAC}{\cos\angle BAC} = \frac{(AB)(AC)\sin\angle BAC}{(AB)(AC)\cos\angle BAC} = \frac{4[ABC]}{AT_B^2 + AT_C^2} = \frac{4\cdot 20}{7^2+11^2} = \boxed{\frac{8}{17}}.$$

6. Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70^\circ$, $\angle ADB = 50^\circ$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$ in degrees.

   Proposed by David Altizio

   Solution. Note that
   $$\angle BAD = 180^\circ - \angle ABD - \angle ADB = 60^\circ,$$
   and thus $\angle PCQ = \angle BCD = 120^\circ$. Furthermore, since $BC = CD$, AC bisects $\angle BAD$. Now let I denote the incenter of $\triangle APQ$. It is well-known that
   $$\angle PIQ = 90^\circ + \frac{\angle PAQ}{2} = 90^\circ + \frac{60^\circ}{2} = 120^\circ,$$
   whence P, C, I, and Q are concyclic. But A, I, and C are collinear, and so in fact $I \equiv C$, i.e. C is the incenter of $\triangle APQ$. From
   $$\angle APD = \angle ABD - \angle BDC = 70^\circ - 30^\circ = 40^\circ \qquad \text{and} \qquad \angle AQB = \angle ADB - \angle DBC = 50^\circ - 30^\circ = 20^\circ,$$
   we thus find that
   $$\angle APQ - \angle AQP = 2(\angle APC - \angle AQC) = 2(40^\circ - 20^\circ) = \boxed{40^\circ}.$$



7. Two non-intersecting circles, ω and Ω, have centers C_ω and C_Ω respectively. It is given that the radius of Ω is strictly larger than the radius of ω. The two common external tangents of Ω and ω intersect at a point P, and an internal tangent of the two circles intersects the common external tangents at X and Y. Suppose that the radius of ω is 4, the circumradius of $\triangle PXY$ is 9, and XY bisects $\overline{PC_\Omega}$. Compute XY.

   Proposed by David Altizio

Solution. The problem statement is equivalent to finding BC, where ABC is a triangle with inradius 4, circumradius 9, and height from A equal to the A-exradius. The following is one such way to do this. Denote by K the area of $\triangle ABC$, s its semiperimeter, r its inradius, R its circumradius, and r_a its A-exradius. Write

$$K = \frac{1}{2}ar_a = r_a(s-a) \qquad \Longrightarrow \qquad a = 2(s-a) = b+c-a.$$

(The first two equalities are well-known formulas for the area of a triangle, where in the first one we substitute r_a for the height from A.) This means that $b+c = 2a$, or $s = \frac{3}{2}a$. Thus, we have $K = rs = 6a$. As a result,

$$6a = \frac{abc}{4R} = \frac{abc}{36} \qquad \Longrightarrow \qquad bc = 216.$$

Now recall that by Heron's Formula,

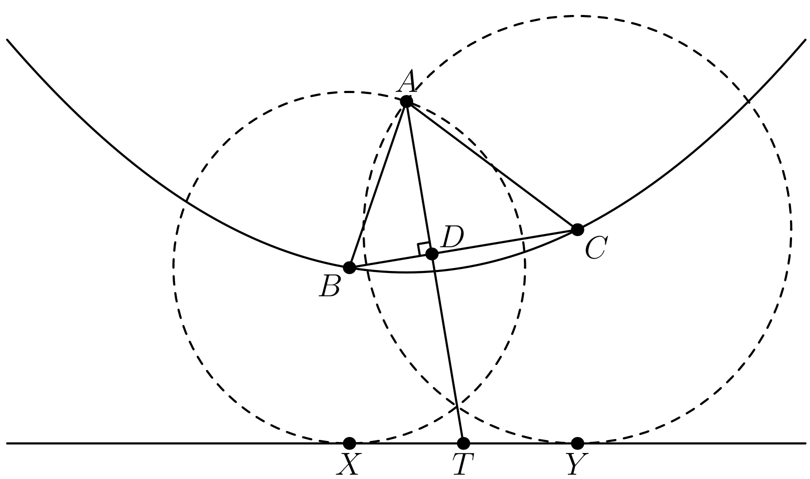
$$\begin{aligned} 6a = \sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{\left(\frac{3}{2}a\right)\left(\frac{1}{2}a\right)(s-b)(s-c)} \\ \Longrightarrow\ 48 = (s-b)(s-c) &= s^2 - s(b+c) + bc \\ &= \left(\frac{3}{2}a\right)^2 - \left(\frac{3}{2}a\right)\left(\frac{1}{2}a\right) + bc = 216 - \frac{3}{4}a^2. \end{aligned}$$

Hence $a = \sqrt{224} = \boxed{4\sqrt{14}}$.

8. In triangle ABC with $AB = 23$, $AC = 27$, and $BC = 20$, let D be the foot of the A altitude. Suppose $\mathcal{P}$ is the parabola with focus A passing through B and C, and denote by T the intersection point of AD with the directrix of $\mathcal{P}$. Determine the value of $DT^2 - DA^2$. (Recall that a parabola $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$.)

Proposed by David Altizio and Evan Chen

Solution. Let ℓ denote the directrix of $\mathcal{P}$, and let X and Y be the projections of B and C respectively onto ℓ. Recall that by definition of a parabola, $AB = BX$ and $AC = CY$. It follows that X is the tangency point of ℓ with the circle ω_B centered at B with radius AB. Similarly, Y is the tangency point of ℓ with the circle ω_C centered at C with radius AC.



Now denote by A' the second intersection point of ω_B and ω_C. Note that $AB = A'B$ and $AC = A'C$, so $\triangle ABC \cong \triangle A'BC$. Thus A' is the reflection of A across BC. Thus A, D, and A' are collinear. It follows that AD is the radical axis of ω_B and ω_C. In particular, T is the midpoint of $\overline{XY}$.

Finally, remark that by Power of a Point,

$$TX^2 = TA' \cdot TA = (TD + AD)(TD - AD) = TD^2 - AD^2.$$

Thus, it suffices to compute TX. This is one half the length of the common external tangent of ω_B and ω_C, which can be easily computed to be

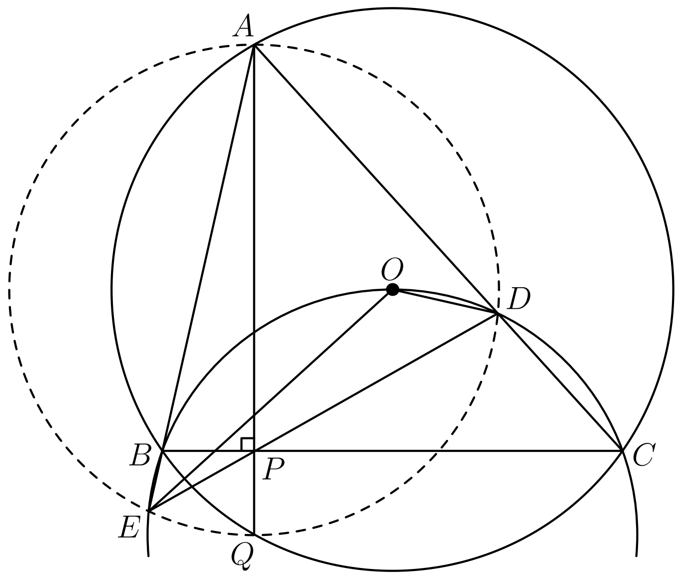
$$\sqrt{BC^2 - (AC - AB)^2} = \sqrt{20^2 - (27 - 23)^2} = 8\sqrt{6}.$$

Thus $TX = 4\sqrt{6}$ and the requested answer is $\boxed{96}$.

9. Let $\triangle ABC$ be an acute triangle with circumcenter O, and let $Q \neq A$ denote the point on $\odot(ABC)$ for which $AQ \perp BC$. The circumcircle of $\triangle BOC$ intersects lines AC and AB for the second time at D and E respectively. Suppose that AQ, BC, and DE are concurrent. If $OD = 3$ and $OE = 7$, compute AQ.

Proposed by David Altizio

Solution. First remark that DE is antiparallel to BC, so $\triangle ADE \sim \triangle ABC$.



Let P be the foot of the perpendicular from A to BC. Note that BC is the radical axis of $\odot(ABC)$ and $\odot(BOC)$ and that DE is the radical axis of $\odot(BOC)$ and $\odot(ADE)$. Hence P is the radical center of all three circles, meaning that AP is the radical axis of $\odot(ABC)$ and $\odot(ADE)$. Since AQ is a chord of $\odot(ABC)$, we may deduce that $ADQE$ is cyclic.

Furthermore, a simple angle chase reveals that

$$\angle ADO = \angle OBC = 90^\circ - \angle A,$$

which implies $DO \perp AB$. Similarly $EO \perp AC$, so O is the orthocenter of $\triangle ADE$. This means that AO and AP are isogonal with respect to $\angle A$. As a result, AQ is a diameter of $\odot(ADE)$, which implies that $ODQE$ is a parallelogram. This means that

$$2(OD^2 + OE^2) = OQ^2 + DE^2 = OA^2 + DE^2.$$

But note that if R' is the circumradius of $\triangle ADE$, then

$$OA^2 + DE^2 = (2R' \cos A)^2 + (2R' \sin A)^2 = 4R'^2,$$

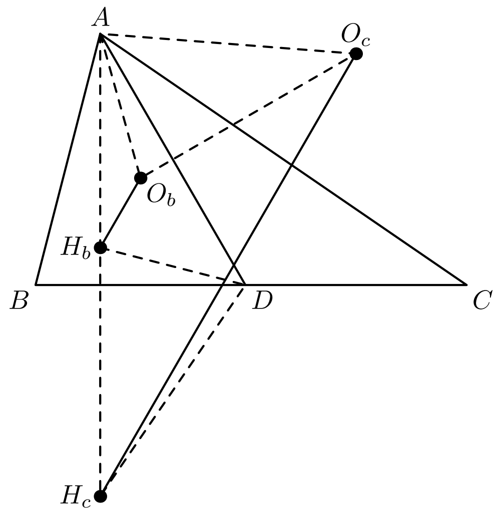
which we know is equal to AQ^2 since AQ is a diameter of $\odot(ADE)$. Thus

$$AQ = \sqrt{2(OD^2 + OE^2)} = \sqrt{2(3^2 + 7^2)} = \boxed{2\sqrt{29}}.$$

10. Suppose $\triangle ABC$ is such that $AB = 13$, $AC = 15$, and $BC = 14$. It is given that there exists a unique point D on side $\overline{BC}$ such that the Euler lines of $\triangle ABD$ and $\triangle ACD$ are parallel. Determine the value of $\frac{BD}{CD}$. (The *Euler* line of a triangle ABC is the line connecting the centroid, circumcenter, and orthocenter of ABC.)

Proposed by David Altizio

Solution. We solve this problem with the configuration shown below; it's not hard to see that this is the only possible one. Here, O_b and O_c are the circumcenters of $\triangle ABD$ and $\triangle ACD$ respectively, while H_b and H_c are the orthocenters of $\triangle ABD$ and $\triangle ACD$ respectively.



We first claim that $\triangle AO_bO_c \sim \triangle ABC \sim \triangle DH_bH_c$. Indeed, these claims are not hard to prove: the first comes from the fact that $\angle AO_bB = \angle AO_cC \implies \triangle AO_bB \sim \triangle AO_cC$, while the second comes from the fact that $DH_b \perp AB$ and $DH_c \perp AC$. Details are left to the interested reader. Furthermore, these triangles are directly similar to each other. Thus, there exists a spiral similarity $\mathcal{S}$ sending $\triangle DH_bH_c \mapsto \triangle AO_bO_c$.

Let $P = H_bH_c \cap O_bO_c$. Then since $H_bO_b \parallel H_cO_c$, we have $PH_b/H_bH_c = PO_b/O_bO_c$. Hence P is the center of spiral similarity sending $\overline{H_aH_b} \mapsto \overline{O_bO_c}$, and thus it must be the center of $\mathcal{S}$. But from the fact that O_bO_c is a perpendicular bisector of $\overline{AD}$, we obtain that

$$\frac{DH_b}{AO_b} = \frac{PD}{PA} = 1,$$

so in fact $\triangle AO_bO_c \cong \triangle DH_bH_c$. Furthermore, if R is the circumradius of $\triangle ABD$, then $R = 2R\cos\angle ABD$, so $\cos\angle ABD = \frac{1}{2}$ and thus $\angle ADB = 60^\circ$.

Now let X be the foot of the altitude from A to BC. Compute $BX = 5$, $CX = 9$, and $AX = 12$. It follows that $DX = 4\sqrt{3}$, and so

$$\frac{BD}{CD} = \boxed{\frac{5+4\sqrt{3}}{9-4\sqrt{3}} = \frac{93+56\sqrt{3}}{33}}.$$