CMIMC 2017
Geometry Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Geometry

1. Let ABC be a triangle with $\angle BAC = 117°$. The angle bisector of $\angle ABC$ intersects side AC at D. Suppose $\triangle ABD \sim \triangle ACB$. Compute the measure of $\angle ABC$, in degrees.

2. Triangle ABC has an obtuse angle at $\angle A$. Points D and E are placed on $\overline{BC}$ in the order B, D, E, C such that $\angle BAD = \angle BCA$ and $\angle CAE = \angle CBA$. If $AB = 10$, $AC = 11$, and $DE = 4$, determine BC.

3. In acute triangle ABC, points D and E are the feet of the angle bisector and altitude from A respectively. Suppose that $AC - AB = 36$ and $DC - DB = 24$. Compute $EC - EB$.

4. Let $\mathcal{S}$ be the sphere with center $(0, 0, 1)$ and radius 1 in $\mathbb{R}^3$. A plane $\mathcal{P}$ is tangent to $\mathcal{S}$ at the point (x_0, y_0, z_0), where x_0, y_0, and z_0 are all positive. Suppose the intersection of plane $\mathcal{P}$ with the xy-plane is the line with equation $2x + y = 10$ in xy-space. What is z_0?

5. Two circles ω_1 and ω_2 are said to be *orthogonal* if they intersect each other at right angles. In other words, for any point P lying on both ω_1 and ω_2, if ℓ_1 is the line tangent to ω_1 at P and ℓ_2 is the line tangent to ω_2 at P, then $\ell_1 \perp \ell_2$. (Two circles which do not intersect are not orthogonal.)

   Let $\triangle ABC$ be a triangle with area 20. Orthogonal circles ω_B and ω_C are drawn with ω_B centered at B and ω_C centered at C. Points T_B and T_C are placed on ω_B and ω_C respectively such that AT_B is tangent to ω_B and AT_C is tangent to ω_C. If $AT_B = 7$ and $AT_C = 11$, what is $\tan \angle BAC$?

6. Cyclic quadrilateral $ABCD$ satisfies $\angle ABD = 70°$, $\angle ADB = 50°$, and $BC = CD$. Suppose AB intersects CD at point P, while AD intersects BC at point Q. Compute $\angle APQ - \angle AQP$.

7. Two non-intersecting circles, ω and Ω, have centers C_ω and C_Ω respectively. It is given that the radius of Ω is strictly larger than the radius of ω. The two common external tangents of Ω and ω intersect at a point P, and an internal tangent of the two circles intersects the common external tangents at X and Y. Suppose that the radius of ω is 4, the circumradius of $\triangle PXY$ is 9, and XY bisects $\overline{PC_\Omega}$. Compute XY.

8. In triangle ABC with $AB = 23$, $AC = 27$, and $BC = 20$, let D be the foot of the A altitude. Let $\mathcal{P}$ be the parabola with focus A passing through B and C, and denote by T the intersection point of AD with the directrix of $\mathcal{P}$. Determine the value of $DT^2 - DA^2$. (Recall that a parabola $\mathcal{P}$ is the set of points which are equidistant from a point, called the *focus* of $\mathcal{P}$, and a line, called the *directrix* of $\mathcal{P}$.)

9. Let $\triangle ABC$ be an acute triangle with circumcenter O, and let $Q \neq A$ denote the point on $\odot(ABC)$ for which $AQ \perp BC$. The circumcircle of $\triangle BOC$ intersects lines AC and AB for the second time at D and E respectively. Suppose that AQ, BC, and DE are concurrent. If $OD = 3$ and $OE = 7$, compute AQ.

10. Suppose $\triangle ABC$ is such that $AB = 13$, $AC = 15$, and $BC = 14$. It is given that there exists a unique point D on side $\overline{BC}$ such that the Euler lines of $\triangle ABD$ and $\triangle ACD$ are parallel. Determine the value of $\frac{BD}{CD}$. (The *Euler* line of a triangle ABC is the line connecting the centroid, circumcenter, and orthocenter of ABC.)