Algebra Tiebreaker

1. Let
$$f(x) = \frac{1}{1 - \frac{1}{1-x}}.$$
Compute $f^{2016}(2016)$, where f is composed upon itself 2016 times.

2. Determine the value of the sum
$$\left| \sum_{1 \le i < j \le 50} ij(-1)^{i+j} \right|.$$

3. Suppose x and y are real numbers which satisfy the system of equations
$$x^2 - 3y^2 = \frac{17}{x} \qquad \text{and} \qquad 3x^2 - y^2 = \frac{23}{y}.$$
Then $x^2 + y^2$ can be written in the form $\sqrt[m]{n}$, where m and n are positive integers and m is as small as possible. Find $m + n$.

Combinatorics Tiebreaker

1. For a set $S \subseteq \mathbb{N}$, define $f(S) = \{\lceil \sqrt{s} \rceil \mid s \in S\}$. Find the number of sets T such that $|f(T)| = 2$ and $f(f(T)) = \{2\}$.

2. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Compute the number of sets of subsets $T = \{A, B, C\}$ with $A, B, C \in S$ such that $A \cup B \cup C = S$, $(A \cap C) \cup (B \cap C) = \emptyset$, and no subset contains two consecutive integers.

3. Let S be the set containing all positive integers whose decimal representations contain only 3s and 7s, have at most 1998 digits, and have at least one digit appear exactly 999 times. If N denotes the number of elements in S, find the remainder when N is divided by 1000.

Computer Science Tiebreaker

1. A *planar* graph is a connected graph that can be drawn on a sphere without edge crossings. Such a drawing will divide the sphere into a number of faces. Let G be a planar graph with 11 vertices of degree 2, 5 vertices of degree 3, and 1 vertex of degree 7. Find the number of faces into which G divides the sphere.

2. The *Stooge sort* is a particularly inefficient recursive sorting algorithm defined as follows: given an array A of size n, we swap the first and last elements if they are out of order; we then (if $n \geq 3$) Stooge sort the first $\lceil \frac{2n}{3} \rceil$ elements, then the last $\lceil \frac{2n}{3} \rceil$, then the first $\lceil \frac{2n}{3} \rceil$ elements again. Given that this runs in $O(n^\alpha)$, where α is minimal, find the value of $(243/32)^\alpha$.

3. Let ε denote the empty string. Given a pair of strings $(A, B) \in \{0,1,2\}^* \times \{0,1\}^*$, we are allowed the following operations:

$$\begin{cases} (A,1) \to (A0,\varepsilon) \\ (A,10) \to (A00,\varepsilon) \\ (A,0B) \to (A0,B) \\ (A,11B) \to (A01,B) \\ (A,100B) \to (A0012,1B) \\ (A,101B) \to (A00122,10B) \end{cases}$$

We perform these operations on (A, B) until we can no longer perform any of them. We then iteratively delete any instance of 20 in A and replace any instance of 21 with 1 until there are no such substrings remaining. Among all binary strings X of size 9, how many different possible outcomes are there for this process performed on (ε, X)?

Geometry Tiebreaker

1. Point A lies on the circumference of a circle Ω with radius 78. Point B is placed such that AB is tangent to the circle and $AB = 65$, while point C is located on Ω such that $BC = 25$. Compute the length of $\overline{AC}$.

2. Identical spherical tennis balls of radius 1 are placed inside a cylindrical container of radius 2 and height 19. Compute the maximum number of tennis balls that can fit entirely inside this container.

3. Triangle ABC satisfies $AB = 28$, $BC = 32$, and $CA = 36$, and M and N are the midpoints of $\overline{AB}$ and $\overline{AC}$ respectively. Let point P be the unique point in the plane ABC such that $\triangle PBM \sim \triangle PNC$. What is AP?

Number Theory Tiebreaker

1. For all integers $n \geq 2$, let $f(n)$ denote the largest positive integer m such that $\sqrt[m]{n}$ is an integer. Evaluate
$$f(2) + f(3) + \cdots + f(100).$$

2. For each integer $n \geq 1$, let S_n be the set of integers $k > n$ such that k divides $30n - 1$. How many elements of the set
$$\mathcal{S} = \bigcup_{i \geq 1} S_i = S_1 \cup S_2 \cup S_3 \cup \ldots$$
are less than 2016?

3. Let $\{x\}$ denote the fractional part of x. For example, $\{5.5\} = 0.5$. Find the smallest prime p such that the inequality
$$\sum_{n=1}^{p^2} \left\{ \frac{n^p}{p^2} \right\} > 2016$$
holds.