

CMMO 2017 Combinatorics Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Combinatorics

1. Robert colors each square in an empty 3 by 3 grid either red or green. Find the number of colorings such that no row or column contains more than one green square.
2. Let S be a subset of $\{1, 2, \dots, 2017\}$ such that for any two distinct elements in S , both their sum and product are not divisible by seven. Compute the maximum number of elements that can be in S .
3. Annie stands at one vertex of a regular hexagon. Every second, she moves independently to one of the two vertices adjacent to her, each with equal probability. Determine the probability that she is at her starting position after ten seconds.
4. At a certain pizzeria, there are five different toppings available and a pizza can be ordered with any (possibly empty) subset of them on it. In how many ways can one order an unordered pair of pizzas such that at most one topping appears on both pizzas and at least one topping appears on neither?
5. Emily draws six dots on a piece of paper such that no three lie on a straight line, then draws a line segment connecting each pair of dots. She then colors five of these segments red. Her coloring is said to be *red-triangle-free* if for every set of three points from her six drawn points there exists an uncolored segment connecting two of the three points. In how many ways can Emily color her drawing such that it is red-triangle-free?
6. Boris plays a game in which he rolls two standard four-sided dice independently and at random, and at the end of the game receives a number of dollars equal to the product of the two rolled numbers. After the initial roll of both dice, however, he can pay two dollars to reroll one die of choice, and he is allowed to pay to reroll as many times as he wishes. If Boris plays to maximize his expected gain, how much money, in dollars, can he expect to win by playing once?
7. Given a finite set $S \subset \mathbb{R}^3$, define $f(S)$ to be the minimum integer k such that there exist k planes that divide \mathbb{R}^3 into a set of regions, where no region contains more than one point in S . Suppose that

$$M(n) = \max\{f(S) : |S| = n\} \text{ and } m(n) = \min\{f(S) : |S| = n\}.$$

Evaluate $M(200) \cdot m(200)$.

8. Andrew generates a finite random sequence $\{a_n\}$ of distinct integers according to the following criteria:
 - $a_0 = 1$, $0 < |a_n| < 7$ for all n , and $a_i \neq a_j$ for all $i < j$.
 - a_{n+1} is selected uniformly at random from the set $\{a_n - 1, a_n + 1, -a_n\}$, conditioned on the above rule. The sequence terminates if no element of the set satisfies the first condition.

For example, if $(a_0, a_1) = (1, 2)$, then a_2 would be chosen from the set $\{-2, 3\}$, each with probability $\frac{1}{2}$. Determine the probability that there exists an integer k such that $a_k = 6$.

9. At a conference, six people place their name badges in a hat, which is shaken up; one badge is then distributed to each person such that each distribution is equally likely. Each turn, every person who does not yet have their own badge finds the person whose badge they have and takes that person's badge. For example, if Alice has Bob's badge and Bob has Charlie's badge, Alice would have Charlie's badge after a turn. Compute the probability that everyone will eventually end up with their own badge.
10. Ryan stands on the bottom-left square of a 2017 by 2017 grid of squares, where each square is colored either black, gray, or white according to the pattern as depicted to the right. Each second he moves either one square up, one square to the right, or both one up and to the right, selecting between these three options uniformly and independently. Noting that he begins on a black square, find the probability that Ryan is still on a black square after 2017 seconds.

