Combinatorics Solutions

1. The phrase "COLORFUL TARTAN" is spelled out with wooden blocks, where blocks of the same letter are indistinguishable. How many ways are there to distribute the blocks among two bags of different color such that neither bag contains more than one of the same letter?

   Proposed by Joshua Siktar

   Solution. Observe that there are five pairs of letters and four singletons. It is not necessary to care about the pairs, since each pair must have one letter in each bag. It then remains to distribute four distinct letters among two distinguishable bags; this can be done in $2^4 = \boxed{16}$ ways.

2. Six people each flip a fair coin. Everyone who flipped tails then flips their coin again. Given that the probability that all the coins are now heads can be expressed as simplified fraction $\frac{m}{n}$, compute $m+n$.

   Proposed by Patrick Lin

   Solution. Observe that each person has a $1-(\frac{1}{2})(\frac{1}{2}) = \frac{3}{4}$ chance of ending up with a head; the only way they do not end heads is if they flip tails twice in a row. Hence with 6 people the probability is $(\frac{3}{4})^6 = \frac{729}{4096}$, and so the answer is $\boxed{4825}$.

3. At CMU, markers come in two colors: blue and orange. Zachary fills a hat randomly with three markers such that each color is chosen with equal probability, then Chase shuffles an additional orange marker into the hat. If Zachary chooses one of the markers in the hat at random and it turns out to be orange, the probability that there is a second orange marker in the hat can be expressed as simplified fraction $\frac{m}{n}$. Compute $m+n$.

   Proposed by Patrick Lin

   Solution. Notice that there is a $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$ chance for the hat to contain 1, 2, 3, and 4 orange markers, respectively, since the original three markers are random and we add one extra orange marker in. Then the probability that we choose an orange marker first is $\frac{1}{8}\cdot\frac{1}{4}+\frac{3}{8}\cdot\frac{2}{4}+\frac{3}{8}\cdot\frac{3}{4}+\frac{1}{8}\cdot\frac{4}{4} = \frac{20}{32}$, and the probability that there is no orange marker left given this is equal to the chance there was only one orange marker given this, which is $\frac{1}{8}\cdot\frac{1}{4} = \frac{1}{32}$. Hence the chance there is another orange marker is given by $1 - \frac{1/32}{20/32} = \frac{19}{20}$, and the answer is thus $\boxed{39}$.

4. Kevin colors three distinct squares in a 3x3 grid. Given that there exist two uncolored squares such that coloring either one of them would create a horizontal or vertical red line, find the number of ways he could have colored the original three squares.

   Proposed by Patrick Lin

   Solution. Observe that, in the original three colored squares, one pair must share the same row and another pair must share the same column in order to guarantee being able to create two different lines. Assume we pick the first square arbitrarily from 9 choices. If we pick the second square sharing the same row or column as the first, we have 4 options, and then we have 4 options for the third. If we pick the second square not sharing either the same row or the same column as the first, we still have 4 options, but then we have only 2 options for the third. This overcounts by a factor of $3! = 6$, and so in total there are $\frac{1}{6}(9\cdot4\cdot4+9\cdot4\cdot2) = \boxed{36}$ ways.

5. Let $\mathcal{S}$ be a regular 18-gon, and for two vertices in $\mathcal{S}$ define the *distance* between them to be the length of the shortest path along the edges of $mathcalS$ between them (e.g. adjacent vertices have distance 1). Find the number of ways to choose three distinct vertices from $\mathcal{S}$ such that no two of them have distance 1, 8, or 9.

   Proposed by Patrick Lin

   Solution. Consider the nine pairs of vertices formed by pairing a vertex with its diametrically opposite vertex. Clearly, no pair can have both its vertices chosen, since they have distance 9. Further, choosing one from the pair is equivalent to choosing the other - a vertex with distance 1 from one will have distance 8 from the other, and vice versa - and hence we can consider the two vertices identical. The problem then reduces to finding the number of ways to choose three distinct vertices in a nonagon, where each vertex represents one such pair in $\mathcal{S}$, such that no two are pairwise adjacent. Choosing the first from 9 vertices, we find there are $\binom{6}{2} - 5 = 10$ ways to choose the remaining two. But we have overcounted by distinguishing the first vertex, and hence there are actually $9 \cdot 10 \cdot \frac{1}{3} = 30$ ways. Now for each choice of three pairs there are $2^3 = 8$ ways to choose the individual vertices, and thus the answer is $8 \cdot 30 = \boxed{240}$.

Thanks to Brice Huang for suggesting an alteration to this problem.

6. Shen, Ling, and Ru each place four slips of paper with their name on it into a bucket. They then play the following game: slips are removed one at a time, and whoever has all of their slips removed first wins. Shen cheats, however, and adds an extra slip of paper into the bucket, and will win when four of his are drawn. Given that the probability that Shen wins can be expressed as simplified fraction $\frac{m}{n}$, compute $m + n$.

   Proposed by Victor Xu, solution by Patrick Lin

   Solution. First, observe that the probability that Ling and Ru win (or lose) are equal. Then

$$2P(\text{L loses}) = P(\text{L loses}) + P(\text{R loses}) = 2P(\text{S wins}) + P(\text{L wins}) + P(\text{R wins}) = 1 + P(\text{S wins}).$$

   It then suffices to compute the probability that Ling loses. Consider continuing to draw slips until all the slips have been drawn. Then if Lings slip is the last one, clearly she loses. If Rus slip is the last one, then Ling loses if the sequence with all instances of R removed ends either with L or with LS. If Shens slip is the last one removed, then the game is entirely symmetric and so everyone has an equal chance to lose. We count probability in each case.

   - Case 1: L last. Given this, she loses with probability 1.
   - Case 2: R last. Given this, the probability she loses is given by $\frac{4}{9} + \frac{5}{9}\frac{4}{8}$.
   - Case 3: S last. Given this, everyone has an equal chance to lose and so her probability of losing is $\frac{2}{3}$.

   Thus the overall probability she loses is

$$\frac{4}{13} + \frac{4}{13}\left(\frac{4}{9} + \frac{5}{9} \cdot \frac{4}{8}\right) + \frac{5}{13} \cdot \frac{2}{3} = \frac{92}{117}.$$

   Hence the desired probability is equal to $2 \cdot \frac{92}{117} - 1 = \frac{67}{117}$, and the answer is $\boxed{184}$.

7. There are eight people, each with their own horse. The horses are arbitrarily arranged in a line from left to right, while the people are lined up in random order to the left of all the horses. One at a time, each person moves rightwards in an attempt to reach their horse. If they encounter a mounted horse on their way to their horse, the mounted horse shouts angrily at the person, who then scurries home immediately. Otherwise, they get to their horse safely and mount it. The expected number of people who have scurried home after all eight people have attempted to reach their horse can be expressed as simplified fraction $\frac{m}{n}$. Find $m + n$.

   Proposed by Patrick Lin

   Solution. We find the expected number of people who mount their horse successfully. Number the horses from left to right $1, 2, \ldots, 8$ and label their owners with the same number. Note that when person i moves, the only way for him to mount his horse is if he is the first of i people (namely, those with labels $1, 2, \ldots, i$) to move. Hence person i mounts his horse with $\frac{1}{i}$ probability. Since we have eight people, the expected number of people who mount their horse is thus $\sum_{i=1}^{8} \frac{1}{i} = H_8 = \frac{761}{280}$. The expected number of people who scurry home is hence

$$8 - \frac{761}{280} = \frac{1479}{280},$$

   and so the answer is $\boxed{1759}$.

8. Brice is eating bowls of rice. He takes a random amount of time $t_1 \in (0,1)$ minutes to consume his first bowl, and every bowl thereafter takes $t_n = t_{n-1} + r_n$ minutes, where t_{n-1} is the time it took him to eat his previous bowl and $r_n \in (0,1)$ is chosen uniformly and randomly. The probability that it takes Brice at least 12 minutes to eat 5 bowls of rice can be expressed as simplified fraction $\frac{m}{n}$. Compute $m+n$.

Proposed by Patrick Lin

Solution. We consider geometric probability in five dimensions. Note that the probability that it takes Brice at least 12 minutes is equal to the probability that he takes at most 3 minutes. Observe that the volume for the figure bounded by $x_1 + x_2 + x_3 + x_4 + x_5 \le 3$ and $x_i > 0$ is a triangular hyperpyramid with side length 3 whose volume is given by $\frac{3^5}{5!}$. But for each i, we have the bound $x_i \le i$, which forms a box with volume $5!$. Now observe that since $3 = 1 + 2$, the volume of the figure outside the box is equal to a similar hyperpyramid with side length 2 (this comes when $x_1 > 1$) and another one with side length 1 (this comes when $x_2 > 2$). There is no other overlap, since it is impossible for both x_1 and x_2 to exceed their respective bounds, and it is impossible for each of the other x_is to exceed their bounds as well. Hence the total volume is given by $\frac{3^5-2^5-1^5}{5!}$. The probability is thus $\frac{3^5-2^5-1^5}{(5!)^2} = \frac{7}{480}$, and so the answer is $\boxed{487}$.

9. 1007 distinct potatoes are chosen independently and randomly from a box of 2016 potatoes numbered $1, 2, \dots, 2016$, with p being the smallest chosen potato. Then, potatoes are drawn one at a time from the remaining 1009 until the first one with value $q < p$ is drawn. If no such q exists, let $S = 1$. Otherwise, let $S = pq$. Then given that the expected value of S can be expressed as simplified fraction $\frac{m}{n}$, compute $m+n$.

Proposed by Patrick Lin

Solution. Note that the largest possible value for p is equal to $(2016 - 1007) + 1 = 1010$. Then for any $1 \le i \le 1010$, the probability that $p = i$ is given by

$$P(p = i) = \frac{\binom{2016-i}{1006}}{\binom{2016}{1007}}.$$

Hence for $i > 1$, q is chosen randomly between 1 and $i-1$, inclusive, and so $E[q \mid p = i] = \frac{p}{2}$. This means $E[S \mid p = i] = \frac{p^2}{2}$. Note that for $i = 1$, we have $E[S \mid p = 1] = \frac{1}{2}$, which abides by the observation for $i > 1$, and so we can just lump everything together. The total expected value is hence

$$E[S] = \sum_{i=1}^{1010} \frac{\binom{2016-i}{1006}}{\binom{2016}{1007}} \cdot \frac{i^2}{2} = \frac{1}{2\binom{2016}{1007}} \sum_{i=1}^{1010} i^2 \binom{2016-i}{1006}.$$

With some application of Hockey Stick, we find that

$$\begin{aligned}
2\binom{2016}{1007} E[S] &= 1^2\binom{2015}{1006} + 2^2\binom{2014}{1006} + \cdots + 1010^2\binom{1006}{1006} \\
&= 1\binom{2016}{1007} + 3\binom{2015}{1007} + \cdots + 2019\binom{1007}{1007} \\
&= 2\left(1\binom{2016}{1007} + 2\binom{2015}{1007} + \cdots + 1010\binom{1007}{1007}\right) - \binom{2017}{1008} \\
&= 2\left(\binom{2017}{1008} + \binom{2016}{1008} + \cdots + \binom{1008}{1008}\right) - \binom{2017}{1008} \\
&= 2\binom{2018}{1009} - \binom{2017}{1008} \\
&= \binom{2018}{1009} + \binom{2017}{1009}.
\end{aligned}$$

Hence we have

$$E[S] = \frac{\binom{2018}{1009} + \binom{2017}{1009}}{2\binom{2016}{1007}} = \frac{\frac{2018!}{1009!1009!} + \frac{2017!}{1009!1008!}}{2\frac{2016!}{1009!1007!}}$$
$$= \frac{3 \cdot 1009 \cdot 2017!}{2 \cdot 1008 \cdot 1009 \cdot 2016!} = \frac{2017}{672},$$

and so the answer is $\boxed{2689}$.

10. For all positive integers $m \geq 1$, denote by $\mathcal{G}_m$ the set of simple graphs with exactly m edges. Find the number of pairs of integers (m, n) with $1 < 2n \leq m \leq 100$ such that there exists a simple graph $G \in \mathcal{G}_m$ satisfying the following property: it is possible to label the edges of G with labels $E_1, E_2, \ldots, E_m$ such that for all $i \neq j$, edges E_i and E_j are adjacent if and only if either $|i - j| \leq n$ or $|i - j| \geq m - n$.

Note: A graph is said to be *simple* if it has no self-loops or multiple edges. In other words, no edge connects a vertex to itself, and the number of edges connecting two distinct vertices is either 0 or 1.

Proposed by David Altizio

Solution. For convenience, we make a few definitions:

- Let f be a function which takes in a graph $G = (V, E)$ and returns another graph $G' = (V', E')$ such that there exists a bijection $g : V' \mapsto E$ with the property that the edge $\{v_1, v_2\}$ is in E' if and only if $g(v_1)$ and $g(v_2)$ are both incident to some common vertex $v \in V$.
- For positive integers m and n with $m \geq 2n$, let $C_{m,n}$ denote the graph with vertex sequence $\{v_i\}_{i=1}^m$ such that vertices v_i and v_j are adjacent iff $|i - j| \leq n$ or $|i - j| \geq m - n$.

Note that the problem statement is equivalent to finding the number of pairs of integers (m, n) such that there exists a graph H with $f(H) = C_{m,n}$.[1] With that in mind, we claim that there are only three possible classes of pairs (m, n) for which an H exists:

- $(m, n) = (i, 1)$ for $2 \leq i \leq 100$;
- $(m, n) = (j, \lfloor \frac{j}{2} \rfloor)$ for $4 \leq j \leq 100$ (note that $j = 2$ and $j = 3$ are already accounted for);
- $(m, n) = (6, 2)$.

This yields $99 + 97 + 1 = \boxed{197}$ possible pairs.

To prove this, we case on the value of n. The cases are ordered by difficulty.

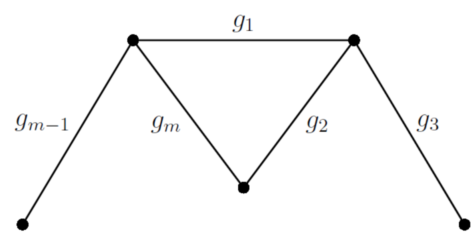
- **CASE 1:** $n = 1$. First consider the case $(m, n) = (2, 1)$. Let G be a path of length 2. It is not hard to show that $f(G) = C_{2,1}$. Hence $m = 2$ works. Otherwise, note that if G is a cycle of length k, then $f(G)$ is also a cycle of length k. (Why?) Hence all cycles of length $k \geq 3$ work, and these are only achieved by $n = 1$. (Note that $C_{3,2}$ is also a cycle of length 3, but this is disallowed by the condition $2n \leq m$.)
- **CASE 2:** $n \geq 3$. I claim that the only conditions that work in this case are cliques. To prove this, we first make an important observation. Assume that $C_{m,n}$ has a clique of size $k > 3$. Then note that all vertices in this clique are connected to each other, meaning that the edges in H associated with these vertices must all touch each other. The only way this can happen is when all these edges are incident to some common vertex. (The case $k = 3$ is special, as we will discuss later; this is the reason why $n = 2$ is a separate case.) Running the reverse logic, it is not hard to show that $f(G)$ has a clique of size at least k iff G has a vertex of degree at least k.

  Suppose that $m > 2n$, i.e. $C_{m,n}$ is not an m-clique. Consider the vertex v_1. Remark that v_{n+2} and v_1 are not connected, and furthermore note that $v_2, v_3, \ldots, v_{n+1}$ are all connected to both of these vertices and to each other. In other words, v_1 through v_{n+1} form a clique, as do v_2 through v_{n+2}. Thus, all edges associated with v_1 through v_{n+2} must be incident to a common vertex, but this is a contradiction since v_1 and v_{n+2} are not adjacent! Hence $C_{m,n}$ must be an m-clique, which forces $(m, n) = (k, \lfloor \frac{k}{2} \rfloor)$. Note that this is constructable for all k by considering a graph G with k edges all incident at a single vertex.

[1] This was the original formulation of the problem; as such, the problem author has decided to re-introduce this notation into the solution in order to minimize as much re-typing as possible.

- **CASE 3:** $n = 2$. In order to tackle this case, we need to explicitly construct a connected subgraph of H in order to derive the contradictions and examples we need.

  Assume $m \geq 7$. Consider vertex v_1, and for convenience let $e_i \equiv g(v_i)$ for all $1 \leq i \leq m$. Note that the edge e_1 is incident to exactly four other edges: e_2, e_3, e_m, and e_{m-1}. I now claim that although e_2 and e_m are connected, they cannot be incident to the same vertex of e_1. To prove this, write $e_1 = \{A, B\}$, and assume WLOG that e_2 and e_m are incident to A. Note that e_3 and e_{m-1} are not connected by the definition of $C_{m,2}$. Thus, these two edges must not both be adjacent to B, meaning that A has degree at least 4. But this is a contradiction, since by the logic in Case 2 $C_{m,2}$ must have a clique of size ≥ 4, which is false. Hence e_2 and e_m are incident to opposite endpoints of e_1. Combined with the fact that e_3 and e_{m-1} are not adjacent, we can conclude that the graph must be of the following form:

  Now it is not hard to derive a contradiction. Consider the edge g_4. Remark that g_4 must be connected to both g_3 and g_2. However, if these three edges were to all share a common vertex, then g_4 would be connected to g_1, which is impossible. Thus, the only placement for g_4 occurs when g_2, g_3, and g_4 form a triangle. But this means that g_4 is connected to g_m, contradiction! Hence $m \geq 7$ is impossible.

  For $m = 4$ and $m = 5$, the construction from Case 2 works, while $m = 6$ works by noting that $f(K_4) = C_{6,2}$. Hence $(4, 2)$, $(5, 2)$, and $(6, 2)$ work as well.

Combining all these cases, we get the three families of solutions listed at the beginning of this solution, and so we are done.