CMIMC 2016

Algebra Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. All answers are integers.
7. If you believe that the test contains an error, submit your protest in writing to Porter 100.

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Algebra

1. In a race, people rode either bicycles with blue wheels or tricycles with tan wheels. Given that 15 more people rode bicycles than tricycles and there were 15 more tan wheels than blue wheels, what is the total number of people who rode in the race?

2. Suppose that some real number x satisfies
$$\log_2 x + \log_8 x + \log_{64} x = \log_x 2 + \log_x 16 + \log_x 128.$$
Given that the value of $\log_2 x + \log_x 2$ can be expressed as $\frac{a\sqrt{b}}{c}$, where a and c are coprime positive integers and b is squarefree, compute abc.

3. Let ℓ be a real number satisfying the equation $\frac{(1+\ell)^2}{1+\ell^2} = \frac{13}{37}$. Then
$$\frac{(1+\ell)^3}{1+\ell^3} = \frac{m}{n},$$
where m and n are positive coprime integers. Find $m+n$.

4. A line with negative slope passing through the point $(18, 8)$ intersects the x and y axes at $(a, 0)$ and $(0, b)$, respectively. What is the smallest possible value of $a + b$?

5. The parabolas $y = x^2 + 15x + 32$ and $x = y^2 + 49y + 593$ meet at one point (x_0, y_0). Find $x_0 + y_0$.

6. For some complex number ω with $|\omega| = 2016$, there is some real $\lambda > 1$ such that ω, ω^2, and $\lambda\omega$ form an equilateral triangle in the complex plane. Then, λ can be written in the form $\frac{a+\sqrt{b}}{c}$, where a, b, and c are positive integers and b is squarefree. Compute $\sqrt{a+b+c}$.

7. Suppose a, b, c, and d are positive real numbers that satisfy the system of equations
$$\begin{aligned}(a+b)(c+d) &= 143,\\ (a+c)(b+d) &= 150,\\ (a+d)(b+c) &= 169.\end{aligned}$$
Compute the smallest possible value of $a^2 + b^2 + c^2 + d^2$.

8. Let r_1, r_2, $\ldots$, r_{20} be the roots of the polynomial $x^{20} - 7x^3 + 1$. If
$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$
can be written in the form $\frac{m}{n}$ where m and n are positive coprime integers, find $m+n$.

9. Let $\lfloor x \rfloor$ denote the greatest integer function and $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Let $1 \leq x_1 < \ldots < x_{100}$ be the 100 smallest values of $x \geq 1$ such that $\sqrt{\lfloor x \rfloor \lfloor x^3 \rfloor} + \sqrt{\{x\}\{x^3\}} = x^2$. Compute
$$\sum_{k=1}^{50} \frac{1}{x_{2k}^2 - x_{2k-1}^2}.$$

10. Denote by $F_0(x)$, $F_1(x)$, $\ldots$ the sequence of Fibonacci polynomials, which satisfy the recurrence $F_0(x) = 1$, $F_1(x) = x$, and $F_n(x) = xF_{n-1}(x) + F_{n-2}(x)$ for all $n \geq 2$. It is given that there exist unique integers λ_0, λ_1, $\ldots$, λ_{1000} such that
$$x^{1000} = \sum_{i=0}^{1000} \lambda_i F_i(x)$$
for all real x. For which integer k is $|\lambda_k|$ maximized?