Algebra Solutions Packet

1. The residents of the local zoo are either rabbits or foxes. The ratio of foxes to rabbits in the zoo is $2:3$. After 10 of the foxes move out of town and half the rabbits move to Rabbitretreat, the ratio of foxes to rabbits is $13:10$. How many animals are left in the zoo?

   Proposed by Monica Pardeshi

   Solution. Let r be the number of rabbits and f the number of foxes originally in the zoo. Then $3f = 2r$ and $\frac{13}{2}r = 10(f-10)$. Solving for f, we have

   $$13r = \frac{39}{2}f = 20f - 200 \implies f = 400.$$

   Substituting back in gives $r = 600$, so the number of animals left is $(400-10) + \frac{600}{2} = \boxed{690}$.

2. For nonzero real numbers x and y, define $x \circ y = \frac{xy}{x+y}$. Compute

   $$2^1 \circ \left(2^2 \circ \left(2^3 \circ \cdots \circ \left(2^{2016} \circ 2^{2017}\right)\right)\right).$$

   Proposed by Patrick Lin

   Solution. Rewrite $x \circ y$ as $\frac{1}{\frac{1}{x}+\frac{1}{y}}$. Now note that for any x, y, z with $xyz \geq 0$,

   $$x \circ (y \circ z) = \frac{1}{\frac{1}{x} + \frac{1}{\frac{1}{\frac{1}{y}+\frac{1}{z}}}} = \frac{1}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}.$$

   Thus the entire expression becomes

   $$\frac{1}{\frac{1}{2}+\frac{1}{2^2}+\cdots+\frac{1}{2^{2017}}} = \boxed{\frac{2^{2017}}{2^{2017}-1}}.$$

3. Suppose $P(x)$ is a quadratic polynomial with integer coefficients satisfying the identity

   $$P(P(x)) - P(x)^2 = x^2 + x + 2016$$

   for all real x. What is $P(1)$?

   Proposed by David Altizio

   Solution. Let $P(x) = ax^2 + bx + c$, so that $P(P(x)) = aP(x)^2 + bP(x) + c$ and

   $$P(P(x)) - P(x)^2 = (a-1)P(x)^2 + bP(x) + c.$$

   Since $\deg P = 2$, $\deg P^2 = 4$, so this expression will be a fourth-degree polynomial unless $a = 1$. Hence $P(x) = x^2 + bx + c$, so the expression above simplifies to

   $$bP(x) + c = b(x^2 + bx + c) + c = bx^2 + b^2x + (bc + c).$$

   From here equating coefficients gives $b = 1$ and $c = 1008$, so $P(x) = x^2 + x + 1008$ and $P(1) = \boxed{1010}$.

4. It is well known that the the special mathematical constant e can be written in the form $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$. With this in mind, determine the value of

   $$\sum_{j=3}^{\infty} \frac{j}{\lfloor \frac{j}{2} \rfloor !}.$$

Express your answer in terms of e.

Proposed by Joshua Siktar

Solution. Write

$$\sum_{j=4}^{\infty} \frac{j}{\lfloor \frac{j}{2} \rfloor !} = \sum_{k=2}^{\infty} \left(\frac{2k}{k!} + \frac{2k+1}{k!} \right) = \sum_{k=2}^{\infty} \frac{4}{(k-1)!} + \sum_{k=2}^{\infty} \frac{1}{k!}.$$

The first sum comes out to $4(e - \frac{1}{0!}) = 4e - 4$, while the second come comes out to $e - \frac{1}{0!} - \frac{1}{1!} = e - 2$. Thus

$$\sum_{j=4}^{\infty} \frac{j}{\lfloor \frac{j}{2} \rfloor !} = (4e-4) + (e-2) = 5e - 6.$$

Adding back the $j = 3$ term (which is $\frac{3}{1!} = 3$) yields a final answer of $\boxed{5e-3}$.

5. The set S of positive real numbers x such that

$$\left\lfloor \frac{2x}{5} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor + 1 = \lfloor x \rfloor$$

can be written as $S = \bigcup_{j=1}^{\infty} I_j$, where the I_i are disjoint intervals of the form $[a_i, b_i) = \{x \mid a_i \le x < b_i\}$ and $b_i \le a_{i+1}$ for all $i \ge 1$. Find $\sum_{i=1}^{2017} (b_i - a_i)$.

Proposed by Andrew Kwon

Solution. Say the disjoint intervals I_j are *funky*. Simple casework yields $[1, \frac{5}{3}), [2, \frac{5}{2}), [3, \frac{10}{3}), [4, 5)$ as the only funky intervals in $[0, 5)$.[1] Furthermore, we note that

$$\left\lfloor \frac{2(x+5)}{5} \right\rfloor + \left\lfloor \frac{3(x+5)}{5} \right\rfloor + 1 = \left\lfloor \frac{2x}{5} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor + 6,$$

and so x is in a funky interval $\Leftrightarrow x+5$ is in a funky interval. Therefore, all funky intervals are translations of the funky intervals found in $[0, 5)$. It is easy to see then that $\sum_{i=1}^{2016} (b_i - a_i) = \frac{5}{2} \cdot \frac{2016}{4} = 1260$, and $b_{2017} - a_{2017} = \frac{2}{3}$. The final answer is $\boxed{\frac{3782}{3}}$.

6. Suppose P is a quintic polynomial with real coefficients with $P(0) = 2$ and $P(1) = 3$ such that $|z| = 1$ whenever z is a complex number satisfying $P(z) = 0$. What is the smallest possible value of $P(2)$ over all such polynomials P?

Proposed by David Altizio

Solution. Note that complex roots of P must come in conjugate pairs. Since the degree of P is odd, P must have one real root, and by the $|z| = 1$ condition this root must be either 1 or -1. However, $P(1) \neq 0$, so -1 must be said root. Now let α, $\bar{\alpha}$, β, and $\bar{\beta}$ be the remaining four roots. (This implicitly covers the real case as well, since it's impossible for one real root of P to be 1 and the other to be -1.) This implies that

$$\begin{aligned} P(z) &= C(z+1)(z-\alpha)(z-\bar{\alpha})(z-\beta)(z-\bar{\beta}) \\ &= C(z+1)(z^2 - (\alpha+\bar{\alpha})z + \alpha\bar{\alpha})(z^2 - (\beta+\bar{\beta})z + \beta\bar{\beta}) \\ &= C(z+1)(z^2 - 2\Re(\alpha)z + 1)(z^2 - 2\Re(\beta)z + 1), \end{aligned}$$

[1] A simple way to perform this casework systematically is as follows: define the function $f : \mathbb{R} \to \mathbb{Z}$ via

$$f(x) = \left\lfloor \frac{2x}{5} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor - \lfloor x \rfloor.$$

Note that this quantity increases by 1 at every multiple of $\frac{5}{2}$ and $\frac{5}{3}$ and decreases by 1 at every integer x. Thus, one can count how many such increases and decreases are made and examine the places at which the function equals one.

where we use $\alpha\bar{\alpha} = |\alpha|^2 = 1$ and similar in the last step. For ease of typesetting, let $a = 2\Re(\alpha)$ and $b = 2\Re(\beta)$, so that $P(z) = C(z+1)(z^2 - az + 1)(z^2 - bz + 1)$ for $|a|, |b| \leq 2$. Plugging in $z = 0$ gives $C = 2$, while plugging in $z = 1$ yields

$$3 = 2 \cdot 2(2-a)(2-b) \quad \Longrightarrow \quad (2-a)(2-b) = \frac{3}{4}.$$

It thus suffices to minimize

$$P(2) = 2 \cdot 3(2^2 - 2a + 1)(2^2 - 2b + 1) = 6(5 - 2a)(5 - 2b)$$

subject to the constraints given above.

Once again, for ease of typesetting set $p = 2 - a$ and $q = 2 - b$. Then $pq = \frac{3}{4}$ and

$$(5 - 2a)(5 - 2b) = (2p + 1)(2q + 1) = 4pq + 2(p + q) + 1 = 4 + 2(p + q).$$

This means that we must minimize $p + q$. Note that since $|a| \leq 2$ and $|b| \leq 2$, p and q are both nonnegative, so we may apply the AM-GM inequality to obtain $p + q \geq 2\sqrt{pq} = \sqrt{3}$. Thus the smallest possible value of $P(2)$ is

$$6(5 - 2a)(5 - 2b) = 6 \cdot [4 + 2(p + q)] = \boxed{24 + 12\sqrt{3}}.$$

Note that equality is achieved via

$$P(z) = 2(z+1)\left(z^2 - \left(2 - \frac{\sqrt{3}}{2}\right)z + 1\right)^2.$$

7. Let a, b, and c be complex numbers satisfying the system of equations

$$\begin{aligned}\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} &= 9,\\ \frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} &= 32,\\ \frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} &= 122.\end{aligned}$$

Find abc.

Proposed by David Altizio

Solution. Let

$$E_r = \frac{a^r}{b+c} + \frac{b^r}{c+a} + \frac{c^r}{a+b}$$

for all nonnegative integers r. Note that

$$\begin{aligned}E_{r+1} + (a^r + b^r + c^r) &= \frac{a^{r+1}}{b+c} + \frac{b^{r+1}}{c+a} + \frac{c^{r+1}}{a+b} + (a^r + b^r + c^r)\\ &= \left(\frac{a^{r+1}}{b+c} + a^r\right) + \left(\frac{b^{r+1}}{c+a} + b^r\right) + \left(\frac{c^{r+1}}{a+b} + c^r\right)\\ &= \frac{a^{r+1} + a^r b + a^r c}{b+c} + \frac{b^{r+1} + b^r c + b^r a}{c+a} + \frac{c^{r+1} + c^r a + c^r b}{a+b}\\ &= (a+b+c)\left(\frac{a^r}{b+c} + \frac{b^r}{c+a} + \frac{c^r}{a+b}\right) = (a+b+c)E_r.\end{aligned}$$

This is this identity that will be the workhorse for our solution.

Note that plugging in $r=1$ gives $32+(a+b+c)=9(a+b+c)$, or $a+b+c=4$. Similarly, note that the $r=2$ case gives $122+(a^2+b^2+c^2)=32(a+b+c)=128 \implies a^2+b^2+c^2=6$. Next, the $r=0$ case yields $9+3=4(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a})$, and so $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=3$. Now write

$$\begin{aligned}
\frac{1}{4-a}+\frac{1}{4-b}+\frac{1}{4-c}&=3\\
\implies (4-a)(4-b)+(4-a)(4-c)+(4-b)(4-c)&=3(4-a)(4-b)(4-c)\\
\implies 48-8(a+b+c)+(ab+bc+ca)&=3(64-16(a+b+c)+4(ab+bc+ca)-abc)\\
&=12(ab+bc+ca)-3abc\\
\implies 11(ab+bc+ca)-16&=3abc.
\end{aligned}$$

Finally, recall that $a+b+c=4$ and $a^2+b^2+c^2=6$ implies $ab+bc+ca=5$, so

$$11(5)-16=39=3abc \quad\implies\quad abc=\boxed{13}.$$

8. Suppose a_1, a_2, …, a_{10} are nonnegative integers such that

$$\sum_{k=1}^{10} a_k = 15 \qquad\text{and}\qquad \sum_{k=1}^{10} ka_k = 80.$$

Let M and m denote the maximum and minimum respectively of $\sum_{k=1}^{10} k^2a_k$. Compute $M-m$.

Proposed by David Altizio

Solution. The key to this problem is the following trick: let m and k be integers between 1 and 10 inclusive. Suppose $(a_{m-1},a_m,a_k,a_{k+1})$ are four elements of a tuple satisfying the given conditions. Replace this tuple with

$$(a_{m-1}-1,a_m+1,a_k+1,a_{k+1}-1).$$

It's easy to see that both equalities are still satisfied, but now

$$\begin{aligned}
&(m-1)^2(a_{m-1}-1)+m^2(a_m+1)+k^2(a_k+1)+(k+1)^2(a_{k+1}-1)\\
&=V+m^2-(m-1)^2+k^2-(k+1)^2\\
&=V+2(m-k)-2,
\end{aligned}$$

where here $V=(m-1)^2a_{m-1}+m^2a_m+k^2a_k+(k+1)^2a_{k+1}$. Hence, as long as $m\le k$, performing such an operation will decrease the value of $\sum_{k=1}^{10} k^2a_k$. Conversely, if $m-k\ge 1$, such an operation will increase the value of the requested quantity.

First we compute m. It is easy to see the minimum value of our expression comes when there exists a j such that only a_j and a_{j+1} are nonzero; otherwise, we could apply this operation with $m-1$ the smallest index k such that $a_k>0$ and $n+1$ the largest such k to decrease $\sum_{k=1}^{10} k^2a_k$ even further. This j must satisfy

$$a_j+a_{j+1}=15 \qquad\text{and}\qquad ja_j+(j+1)a_{j+1}=80.$$

Note that the second equation becomes

$$j(a_j+a_{j+1})+a_{j+1}=15j+a_{j+1}=80.$$

Now remark that by integer bounding the only possible value of j is $j=5$, which gives $a_{j+1}=5$. Hence $a_5=10$ and $a_6=5$, so

$$m=5^2\cdot 10+6^2\cdot 5=430.$$

Computing M is similar, but the required conditions are a bit trickier. First remark that the system of equations

$$\begin{cases} a_1+a_{10} &=15,\\ a_1+10a_{10} &=80 \end{cases}$$

has unique solution $(a_1, a_{10}) = (\frac{70}{9}, \frac{65}{9})$; these are not integers, and as such it is impossible for only a_1 and a_{10} to be nonzero. With this in mind, we claim that the sum is minimized under the condition that

$$a_2 + a_3 + \cdots + a_9 = 1;$$

in other words, exactly one of these numbers is 1 and the rest are zeros. To see this, suppose the contrary. Write each of a_2 through a_9 as a sum of 1s (so for example, $2 = 1 + 1$). Pick two of these ones, supposing they come from a_j and a_k with $j \le k$. Now by repeatedly applying the operation

$$(0, 1, \ldots, 1, 0) \mapsto (1, 0, \ldots, 0, 1),$$

we can force at least one of these ones out toward the edges to either a_1 or a_{10}. This means that the quantity $a_2 + \cdots + a_9$ decreases by at least one. The claim follows by an inductive argument on this quantity.

As such, in order for the maximum to be achieved, we need

$$\begin{cases} a_1 + a_{10} &= 14, \\ a_1 + 10a_{10} &= 80 - k \end{cases}$$

for some integer $2 \le k \le 9$. Subtracting the equations and taking mod 9 yields

$$0 \equiv 9a_{10} \equiv 66 - k \equiv 3 - k \pmod 9 \quad \Longrightarrow \quad k = 3.$$

Now solving the resulting system gives $(a_1, a_{10}) = (7, 7)$, so

$$M = 1^2 \cdot 7 + 3^2 \cdot 1 + 10^2 \cdot 7 = 716$$

and the requested answer is $716 - 430 = \boxed{286}$.

9. Define a sequence $\{a_n\}_{n=1}^{\infty}$ via $a_1 = 1$ and $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$ for all $n \ge 1$. What is the smallest N such that $a_N > 2017$?

   Proposed by Andrew Kwon

   Solution. We first claim that all powers of 4 appear in this sequence, and that these are the only perfect squares in this sequence. Evidently $a_1 = 1, a_4 = 4$, and so the claim is not false yet.

   In general, for $k \ge 2$ suppose $a_k = n^2 + r$ with $1 \le r \le n$.[2] Then, $a_{k+2} = n^2 + 2n + r = (n+1)^2 + (r-1)$, and inductively we find $a_{k+2r} = (n+r)^2$. Furthermore, none of the terms between a_k, a_{k+2r} are perfect squares. In particular, if $a_{k-1} = n^2$, then $a_k = n^2 + n$ and $a_{k+2n} = 4n^2$. As we have verified that the first perfect squares in our sequence are 1 and 4, the only perfect squares in our sequence are powers of 4.

   It is not hard to see that $\lfloor \sqrt{a_n} \rfloor$ will attain all positive integer values, but we claim that it will attain powers of 2 three times, and all other values twice. Indeed, if $n^2 + n \le a_k \le n^2 + 2n$ for some n, then we must have $n^2 \le a_{k-1} \le n^2 + n$, and so $a_{k-1}, a_k \in [n^2, (n+1)^2)$. This corresponds to $\lfloor \sqrt{a_{k-1}} \rfloor, \lfloor \sqrt{a_k} \rfloor = n$. The only way for three terms a_{k-1}, a_k, a_{k+1} to be in the interval $[n^2, (n+1)^2)$ is if $a_{k-1} = n^2, a_k = n^2 + n$, and $a_{k+1} = n^2 + 2n$. This is precisely when $\lfloor \sqrt{a_{k-1}} \rfloor, \lfloor \sqrt{a_k} \rfloor, \lfloor \sqrt{a_{k+1}} \rfloor$ are powers of 2.

   Now we proceed by consideration of adding consecutive differences. We consider

$$a_N = 2(1 + 2 + \ldots + k) + (1 + 2 + \ldots + 2^{\ell-1}) > 2017$$

   or

$$a_N = 2(1 + 2 + \ldots + k) + (1 + 2 + \ldots + 2^{\ell}) > 2017,$$

   where ℓ is the unique integer such that $2^\ell \le k < 2^{\ell+1}$ and we add $1 + \ldots + 2^{\ell-1}$ or $1 + \ldots + 2^\ell$ because those differences appear three times rather than twice, but we do not yet know whether the third contribution

[2] We need not seriously consider the case $n + 1 \le r \le 2n$, as $a_{k+1} = n^2 + n + r$, and when $1 \le r \le n$ we have $n + 1 \le n + r \le 2n$.

of 2^ℓ is necessary or not. Now the above expressions are equivalent to k^2+k+2^ℓ and $k^2+k+2^{\ell+1}$. As $43\cdot 44 = 1892, 44\cdot 45 = 1980$ we find $k=44$ suffices to guarantee $a_N = 2044 > 2017$ when we include $2^\ell = 32$. To determine the value of N, we use the fact that we have added $2\cdot 44+6$ consecutive differences, and so cumulatively we have calculated the 95^{th} term of the sequence, and $N = \boxed{95}$ is minimal.

10. Let c denote the largest possible real number such that there exists a nonconstant polynomial P with

$$P(z^2) = P(z-c)P(z+c)$$

for all z. Compute the sum of all values of $P(\frac{1}{3})$ over all nonconstant polynomials P satisfying the above constraint for this c.

Proposed by David Altizio

Solution. We claim that $c = \frac{1}{2}$.

First note that if α is a root of P, then plugging in $z = \alpha + c$ yields

$$P((\alpha+c)^2) = P(\alpha)P(\alpha+2c) = 0,$$

so that $(\alpha+c)^2$ is a root of P as well. Similarly, $(\alpha-c)^2$ must also be a root of P.

Now suppose $c > \frac{1}{2}$, and let z be a possible root of P. Define a sequence of complex numbers $\{z_k\}_{k=0}^\infty$ such that $z_0 = z$ and such that z_{k+1} is either equal to $(z_k+c)^2$ or $(z_k-c)^2$. I claim it is always possible to choose a sequence with the property that the sequence $\{|z_k|\}_{k=0}^\infty$ is strictly increasing. To see this, recall by the Parallelogram Law,

$$|z-c|^2 + |z+c|^2 = 2(|z|^2+c^2).$$

It thus follows that one of $|z-c|^2$ and $|z+c|^2$ must be at least $|z|^2+c^2$ (else the entire sum would be too small), so we can choose z_{k+1} such that $|z_{k+1}| \geq |z_k|^2 + c^2$. But note that

$$|z|^2 + c^2 > |z| \quad\iff\quad \left(|z| - \frac{1}{2}\right)^2 + c^2 > \frac{1}{4},$$

which is always true for $c > \frac{1}{2}$. Thus $|z_{k+1}| > |z_k|$, as desired. It follows that $\{z_k\}_{k=0}^\infty$ is an infinite sequence of roots of P, which is a contradiction.

It suffices to classify all polynomials satisfying the equation when $c = \frac{1}{2}$. To do this, remark that there are two equality cases in the above analysis. The first occurs in the choice of z_{k+1}; equality here occurs when $|z-c|^2 = |z+c|^2$, or when z is purely imaginary. The second equality case occurs in completing the square. For $c = \frac{1}{2}$, we need $(|z| - \frac{1}{2})^2 = 0$, i.e. $|z| = \frac{1}{2}$. It follows that $\frac{1}{2}i$ and $-\frac{1}{2}i$ are the only possible roots of P, and furthermore it is easy to see that these roots must occur with equal multiplicity. Indeed, taking $P(z) = z^2 + \frac{1}{4}$, we see that

$$\begin{aligned} P\left(z-\frac{1}{2}\right)P\left(z+\frac{1}{2}\right) &= \left(\left(z-\frac{1}{2}\right)^2 + \frac{1}{4}\right)\left(\left(z+\frac{1}{2}\right)^2+\frac{1}{4}\right) \\ &= \left(z^2 - z + \frac{1}{2}\right)\left(z^2+z+\frac{1}{2}\right) \\ &= \left(z^2+\frac{1}{2}\right)^2 - z^2 = z^4 + \frac{1}{4} = P(z^2). \end{aligned}$$

Hence $P(z) = (z^2+\frac{1}{4})^n$ for some integer $n \geq 1$, and it follows that the sum of all possible values of $P(\frac{1}{3})$ is

$$\sum_{n\geq 1}\left(\frac{1}{9}+\frac{1}{4}\right)^n = \sum_{n\geq 1}\left(\frac{13}{36}\right)^n = \boxed{\frac{13}{23}}.$$