CMIMC 2017
Algebra Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 10 short-answer problems to be solved in 60 minutes. Each question is worth one point.
3. Write your name, team name, and team ID on your answer sheet. Circle the subject of the test you are currently taking.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. Answers must be reasonably simplified.
7. If you believe that the test contains an error, submit your protest in writing to Doherty 2302 by the end of lunch.

Algebra

1. The residents of the local zoo are either rabbits or foxes. The ratio of foxes to rabbits in the zoo is $2:3$. After 10 of the foxes move out of town and half the rabbits move to Rabbitretreat, the ratio of foxes to rabbits is $13:10$. How many animals are left in the zoo?

2. For nonzero real numbers x and y, define $x \circ y = \frac{xy}{x+y}$. Compute
$$2^1 \circ \left(2^2 \circ \left(2^3 \circ \cdots \circ \left(2^{2016} \circ 2^{2017}\right)\right)\right).$$

3. Suppose $P(x)$ is a quadratic polynomial with integer coefficients satisfying the identity
$$P(P(x)) - P(x)^2 = x^2 + x + 2016$$
for all real x. What is $P(1)$?

4. It is well known that the mathematical constant e can be written in the form $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \cdots$. With this in mind, determine the value of
$$\sum_{j=3}^{\infty} \frac{j}{\lfloor \frac{j}{2} \rfloor !}.$$
Express your answer in terms of e.

5. The set S of positive real numbers x such that
$$\left\lfloor \frac{2x}{5} \right\rfloor + \left\lfloor \frac{3x}{5} \right\rfloor + 1 = \lfloor x \rfloor$$
can be written as $S = \bigcup_{j=1}^{\infty} I_j$, where the I_i are disjoint intervals of the form $[a_i, b_i) = \{x \mid a_i \le x < b_i\}$ and $b_i \le a_{i+1}$ for all $i \ge 1$. Find $\sum_{i=1}^{2017} (b_i - a_i)$.

6. Suppose P is a quintic polynomial with real coefficients with $P(0) = 2$ and $P(1) = 3$ such that $|z| = 1$ whenever z is a complex number satisfying $P(z) = 0$. What is the smallest possible value of $P(2)$ over all such polynomials P?

7. Let a, b, and c be complex numbers satisfying the system of equations
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 9,$$
$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} = 32,$$
$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} = 122.$$
Find abc.

8. Suppose $a_1, a_2, \ldots, a_{10}$ are nonnegative integers such that
$$\sum_{k=1}^{10} a_k = 15 \qquad \text{and} \qquad \sum_{k=1}^{10} k a_k = 80.$$
Let M and m denote the maximum and minimum respectively of $\sum_{k=1}^{10} k^2 a_k$. Compute $M - m$.

9. Define a sequence $\{a_n\}_{n=1}^{\infty}$ via $a_1 = 1$ and $a_{n+1} = a_n + \lfloor \sqrt{a_n} \rfloor$ for all $n \ge 1$. What is the smallest N such that $a_N > 2017$?

10. Let c denote the largest possible real number such that there exists a nonconstant polynomial P with
$$P(z^2) = P(z-c)P(z+c)$$
for all z. Compute the sum of all values of $P(\frac{1}{3})$ over all nonconstant polynomials P satisfying the above constraint for this c.